Wave kinetics with the account for finite non-Gaussianity effects

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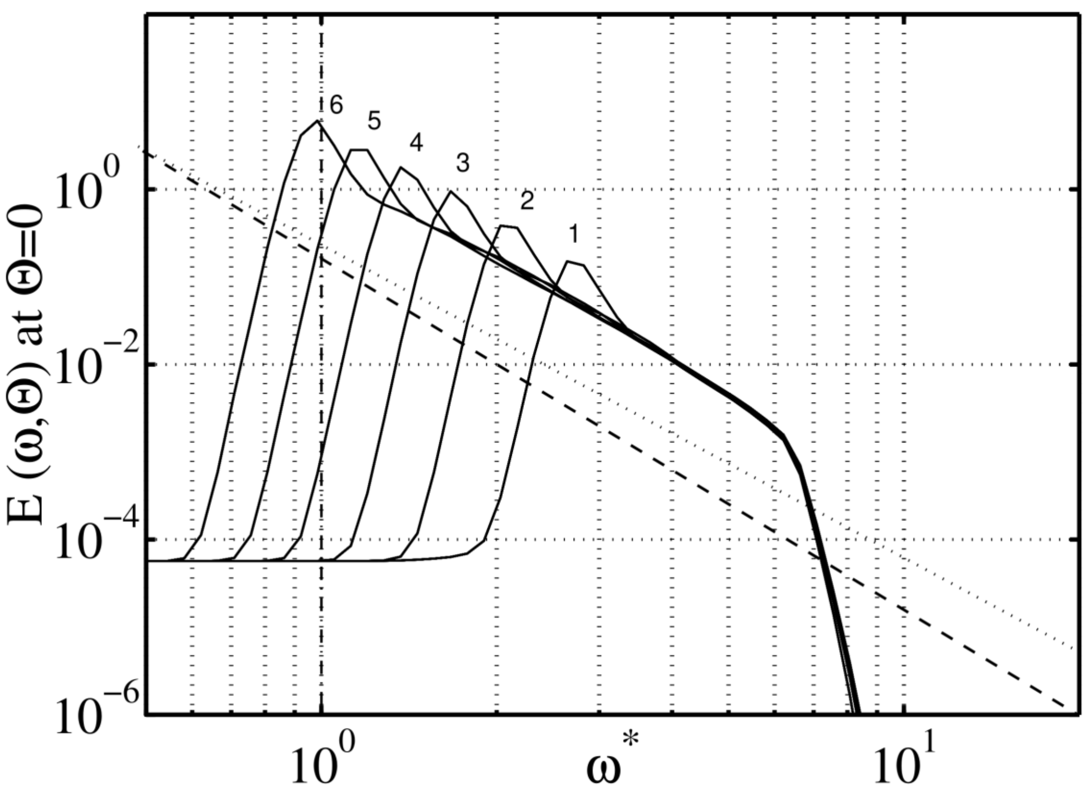
wave kinetics and the Hasselmann equation

- The Hasselmann equation (KE) is the basis of virtually all wind-wave modelling
- There is a consensus that the KE does capture main features of wave field evolution
- The equation is based on the firm foundation of the theory of wave turbulence, which is well-established (assumptions are mild; statistical closure is natural; logic is rock solid)
- The nontrivial additional assumption under the KE is that of quasi-stationarity, so strictly speaking it is not applicable for non-stationary cases (e.g. rapid changes of wind)
- Under stationary conditions, the equation predicts strict self-similarity of spectral development



Wind wave spectral development as numerical solution of the Hasselmann equation. Figure from Badulin, Pushkarev, Resio & Zakharov (2005)

what's beyond the Hasselmann equation?

The Hasselmann equation has the form

$$\frac{\partial n_0}{\partial t} = 4\pi \int T_{0123}^2 \, f_{0123} \delta_{0+1-2-3} \delta(\Delta\omega) \mathrm{d}\mathbf{k}_{123}, \tag{1}$$

where $f_{0123} = n_2 n_3 (n_0 + n_1) - n_0 n_1 (n_2 + n_3)$.

Janssen (2003) highlighted the point about the stationarity assumption and derived a modification that does not use the large-time limit (but assumes slow spectral amplitudes)

$$\frac{\partial n_0}{\partial t} = 4 \int T_{0123}^2 \, f_{0123} \delta_{0+1-2-3} R_i(\Delta\omega, t) \mathrm{d}\mathbf{k}_{123}, \tag{2}$$

where $G = \mathrm{i} \int_0^t e^{\mathrm{i}\Delta\omega(\tau - t)} \mathrm{d}\tau = R_r + \mathrm{i}R_i$, so that $R_i(\Delta\omega, t) = \frac{\sin(\Delta\omega t)}{\Delta\omega}$. For large time, $\lim_{t\to\infty} R_i(\Delta\omega, t) = \pi\delta(\Delta\omega)$, and the Hasselmann equation is recovered.

Janssen's equation is much easier to simulate numerically (although there are many more interactions, it is easy to process them in a parallel computation).

generalised kinetic equation (gKE)

Annenkov & Shrira (2006) removed the requirement of slow amplitudes evolution, and obtained the generalised kinetic equation (gKE)

$$\frac{\partial n_0}{\partial t} = 4\mathrm{Re}\int \left\{ T_{0123}^2 \left[\int_0^t e^{-\mathrm{i}\Delta\omega(\tau - t)} f_{0123} \mathrm{d}\tau \right] - \frac{\mathrm{i}}{2} T_{0123} J_{0123}(0) e^{\Delta\omega t} \right\} \delta_{0+1-2-3} \mathrm{d}\mathbf{k}_{123}\,. \quad (3)$$

Here $J_{0123}(0)$ is the initial value of the four-wave cumulant $J_{0123}(t)$ at the start of the evolution (usually taken as zero, i.e. "cold start").

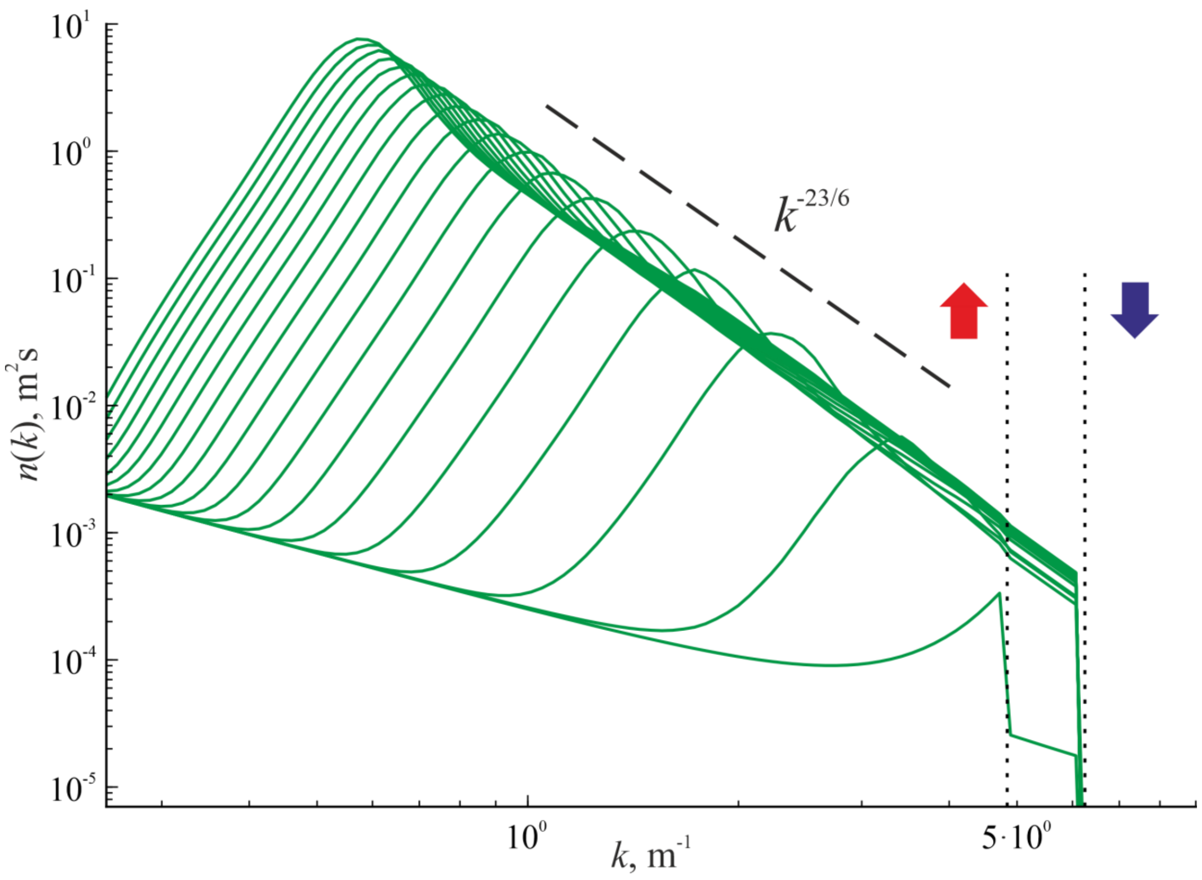
The equation is non-local in time. In practice, it is convenient to use the current value of $J_{0123}(t)$ as the new initial condition (so that the non-locality in time is limited to one step).

This equation now explicitly involves correlators (that is, at least N^2 oscillating functions for N spectral amplitudes n_j).

It is also easily parallelisable, but correlators are sensitive to accumulating numerical noise in higher frequences, leading to numerical instability.

gKE and the Hasselmann equation

- For large times and stationary conditions, all equations tend to the Hasselmann equation
- Although the gKE is derived without the stationarity assumption, in reality it does not show much difference even for non-stationary cases (e.g. rapid changes of wind)
- For large times, all equations predict strict self-similarity of spectral development, and permanent spectral shape



Evolution of a wind wave field under constant wind forcing, *U*/*c*=2.0, code provided by Gerbrant van Vledder

observational issues

- This is not what is observed. It is well-known from measurements that the spectral shape of wind waves depends on the stage of wave development
- For example, cf Donelan et al (1985) parameterisation (figure on the right)
- Young waves have relatively large spectral peakedness (JONSWAP peakedness parameter γ around or above 3)
- As waves mature, γ goes through a transition to near 1 at U/c (inverse wave age) near 1
- For even older waves, γ settles to just below 1
- Fully developed waves have Pierson-Moskowitz (PM) spectral shape, which is well established statistically (e.g. Babanin & Soloviev 1998)

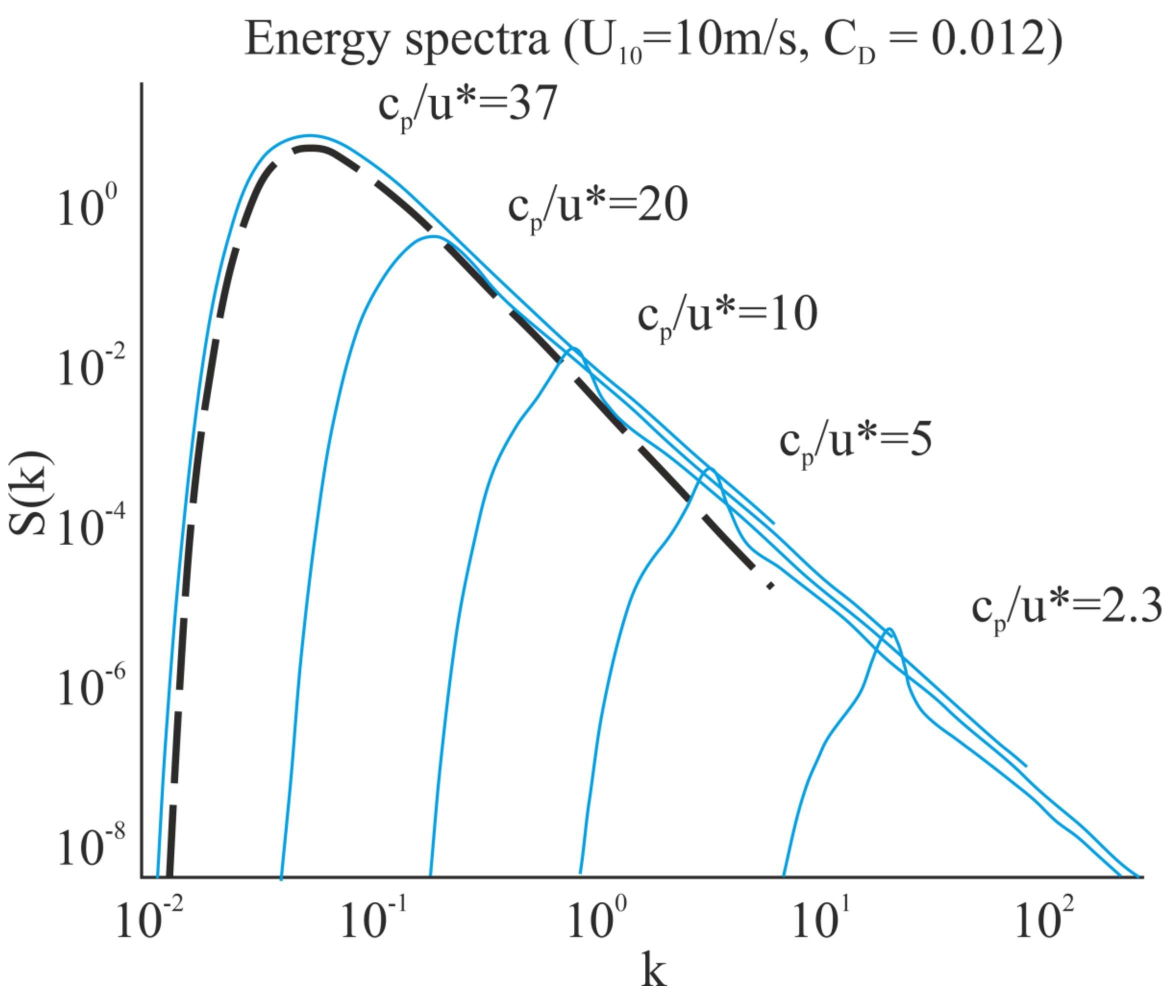
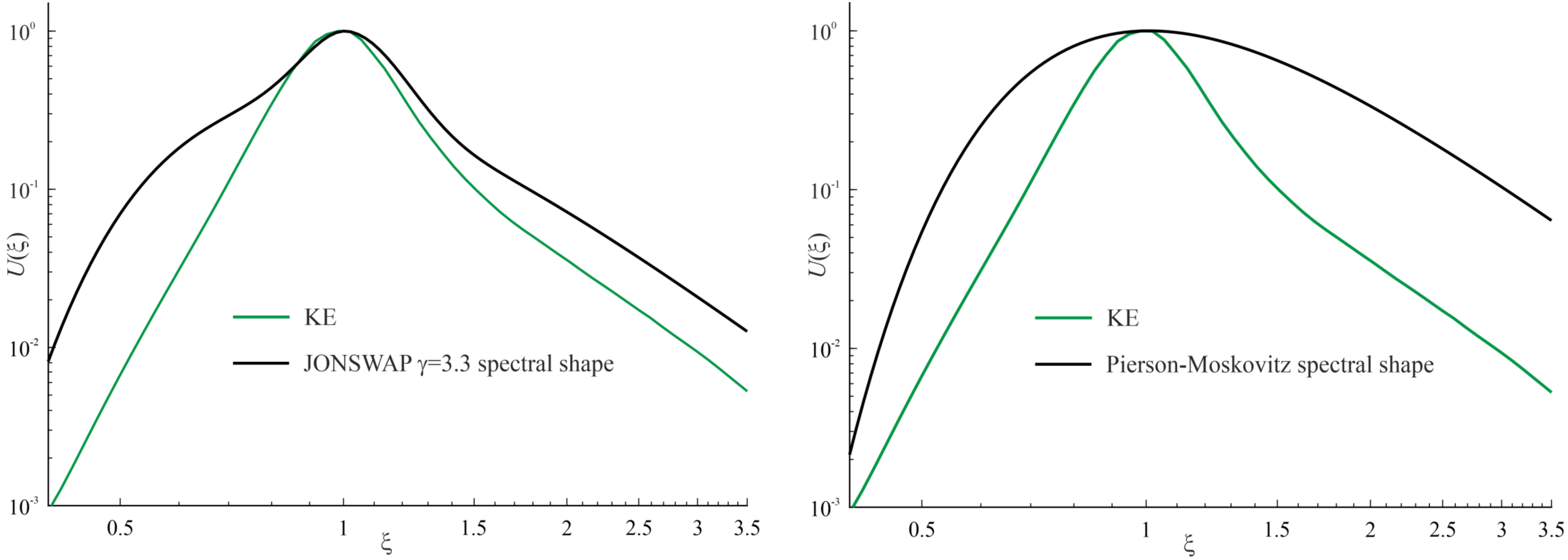


Figure from Kalmikov (2010)

JONSWAP and Pierson-Moskowitz spectral shapes

The self-similar shape of the Hasselmann equation solutions is rather far from the known parameterisations of wind wave spectra, which also are not known to have universal shape. Although *the original JONSWAP spectral shape was assumed to be self-similar… treatment of γ as a constant, as Hasselmann et al (1973) did, is clearly unsuitable* (Babanin & Soloviev 1998).



Comparison of KE self-similar solutions with moderately young (JONSWAP γ=3.3) and mature (Pierson-Moskowitz) spectral shapes

theoretical issues

- The Hasselmann equation

$$\frac{\partial n}{\partial t} = 4\pi \int T_{0123}^2 f_{0123} \delta_{0123} \delta(\Delta\omega) \mathrm{d}k_{0123}$$

where $f = n_2 n_3 (n_0 + n_1) - n_0 n_1 (n_2 + n_2)$, is a real equation, homogeneous with respect to spectrum. It has strict ε^6 ("kinetic") scaling of growth rates, ε^{-4} timescale, and exactly the same spectral shape for all levels of nonlinearity (even infinitesimal). The downshift rate has a simple rescaling law (Zakharov *et al* 2015)

- But this cannot be right. Nonlinear dynamical systems do not behave in the same way for *infinitely small* nonlinearity, and for *small but finite* nonlinearity. Generally speaking, there must be some finite nonlinearity effects, which may lead to $O(1)$ deviations of some characteristics
- Earlier we put forward a hypothesis that for out-of-equilibrium situations there must be ε^4 ("dynamic") scaling of growth rates, and faster-than-kinetic adjustment towards equilibrium
- We have indeed found (with DNS) the ε^4 scaling. But we did not find any faster adjustment.

it’s Gaussianity, not “quasi-Gaussianity”

- Since the gKE (and the derivation of the Hasselmann equation) explicitly involves four-wave cumulants, it is often said that the “quasi-Gaussian statistical closure” is applied
- But actually the closure applied is representing the 6^{th} order correlator in terms of 2^{nd} order correlators only. This is Gaussianity, not “quasi-Gaussianity”
- “Quasi-Gaussian closure” would only drop the cumulant, but retain the 2^{nd} and 4^{th} order terms in the 6^{th} order correlator expansion
- Incidentally, this is seen from the fact that the Hasselmann equation can be derived without the use of correlators, assuming random phases (Onorato & Dematteis 2020)
- The standard closure neglects all finite non-Gaussianity effects
- But this means that we cannot hope to get finite nonlinearity effects either
- Wave kinetics based on the Hasselmann equation and its generalisations shows us what happens in the limit of infinitely small nonlinearity, magnified.

wave kinetics with the account for finite non-Gaussianity

The starting point is the Zakharov equation for complex amplitudes

$$\mathrm{i}\frac{\partial b_0}{\partial t} = \omega_0 b_0 + \int T_{0123} b_1^* b_2 b_3 \delta_{0+1-2-3}\, \mathrm{d}\mathbf{k}_{123}.$$

We assume that there are no wave interactions other than 2↔2.

The statistical description can be obtained in terms of correlators of $b(\boldsymbol{k}, t)$ as

$$\frac{\partial n_0}{\partial t} = 2\mathrm{Im}\int T_{0123} J_{0123} \delta_{0+1-2-3}\, \mathrm{d}\mathbf{k}_{123}. \qquad (4)$$

Here, n_0 is the 2nd order correlator, $\langle b_0 b_1 \rangle = n_0 \delta_{0-1}$, J_{0123} is the 4th order cumulant.

In the next order

$$\left(\frac{\partial}{\partial t} - \mathrm{i}\Delta\omega\right) J_{0123} = 2\mathrm{i}\int \{T_{0456}\delta_{0+4-5-6} I_{156234} + T_{1456}\delta_{1+4-5-6} I_{056234} - T_{2456}\delta_{2+4-5-6} I_{014356} - T_{3456}\delta_{3+4-5-6} I_{014256}\}\, \mathrm{d}\mathbf{k}_{456} \qquad (5)$$

where I_{ijkmnl} are 6th order correlators. System (4-5) is closed (because there are no higher-order interactions, I_{ijkmnl} can be expressed in terms of lower-order correlators).

"exact" wave kinetic theory

Equations (4-5) are also exact, within the (mild) assumptions underlying the wave turbulence theory (cf Newell & Rumph 2013). The crucial point is how to represent I_{ijkmnl}. Formally, we must write 6th order I_{ijkmnl} in terms of 2nd and 4th order correlators. Then, Eq. (5) becomes

$$\left(\frac{\partial}{\partial t} - \mathrm{i}\Delta\omega\right) J_{0123} = 2\mathrm{i} T_{0123} f_{0123} + 2\mathrm{i}\left(\hat{L}J\right)_{0123},$$

where $f_{0123} = n_2 n_3 (n_0 + n_1) - n_0 n_1 (n_2 + n_3)$, and $\hat{L}J$ is a linear operator

$$\left(\hat{L}J\right)_{0123} = M_{0123} + M_{1023} - M_{2301} - M_{3201},$$

$$M_{0123} = n_1 \int T_{0145} J_{4523} \delta_{0145} \mathrm{d}\boldsymbol{k}_{45} - n_2 \int T_{0425} J_{1534} \delta_{0425} \mathrm{d}\boldsymbol{k}_{45} - n_3 \int T_{0435} J_{1524} \delta_{0435} \mathrm{d}\boldsymbol{k}_{45}$$

These expressions were derived by Zakharov 1999 (Eur. J. Mech.)

"exact" wave kinetic theory - formulation

We don't have a single "wave kinetic equation" here, but a system of two equations (**gKE+**):

$$\frac{\partial n_0}{\partial t} = 2\mathrm{Im} \int T_{0123} J_{0123} \delta_{0+1-2-3} \, \mathrm{d}\mathbf{k}_{123}, \qquad (6a)$$

$$\left(\frac{\partial}{\partial t} - \mathrm{i}\Delta\omega\right) J_{0123} = 2\mathrm{i} T_{0123} f_{0123} + 2\mathrm{i}\left(\hat{L}J\right)_{0123}. \qquad (6b)$$

The first equation for n_j has N degrees of freedom (the number of spectral densities).
The second equation operates on correlators (and has roughly N^2 degrees of freedom).

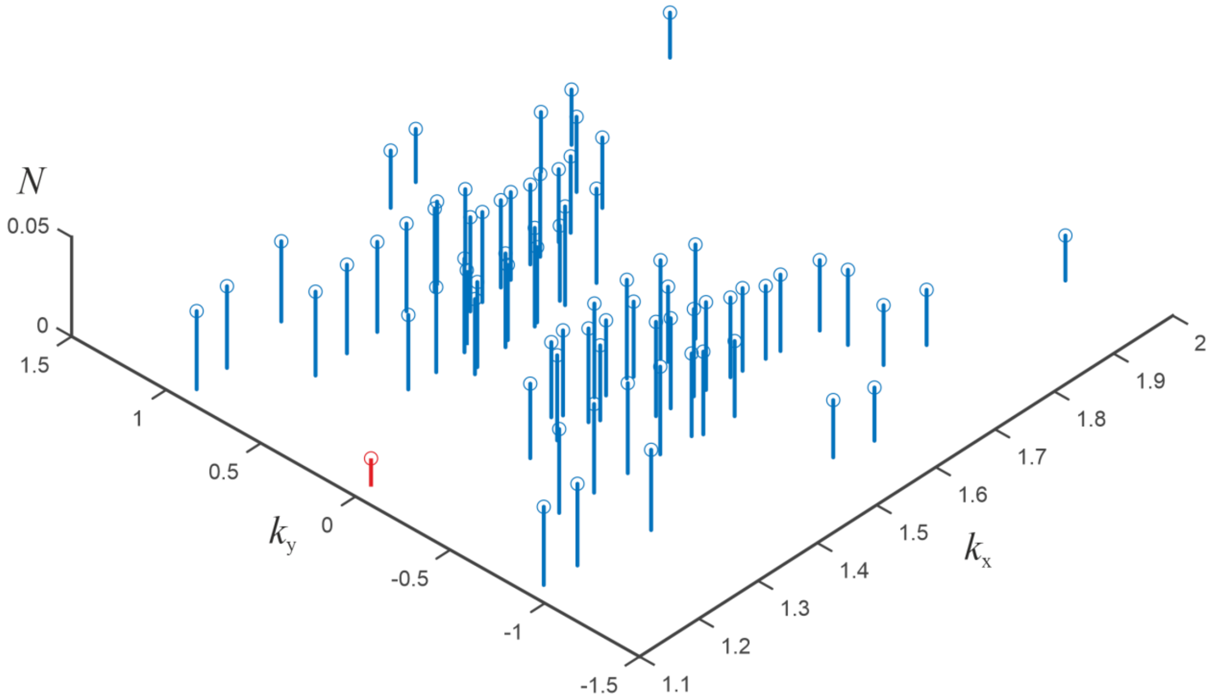
This is a rather refreshing picture: waves interact (in quartets), but **correlators also interact** (in triplets).
Example: three quartets $\mathbf{k}_0 + \mathbf{k}_1 = \mathbf{k}_2 + \mathbf{k}_3 = \mathbf{k}_4 + \mathbf{k}_5$. The three correlators are linked by a linear equation.

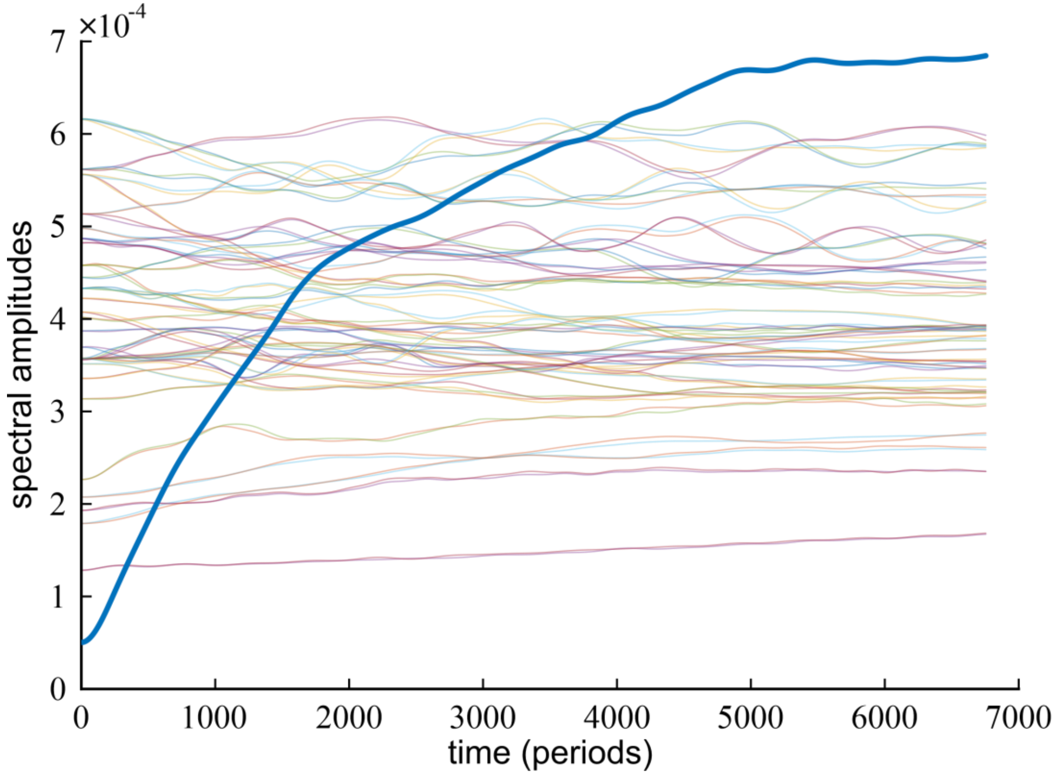
System (6*ab*) is to be solved numerically.

what is the price of neglecting $\hat{L}J$?

- Not clear *a priori*
- It is in the next order, but it stands on the right-hand side of the evolution equation for correlators. Neglecting it can be justified up to a certain timescale only
- In that equation, we have spectral amplitudes n_j as coefficients. They become much larger around the spectral peak
- Strictly speaking, this term is dropped not because there are valid arguments to drop it, but just because the problem with it is far too complicated. Basically, we drop this term not because we have the right to, but because we want to obtain a kinetic equation in a closed form (evolution of spectrum in terms of spectrum), and the term is in the way.
- Zakharov (1999): "It is still very complicated... one has to neglect $\hat{L}J$"
- Neglecting $\hat{L}J$ means omitting the effects of finite non-Gaussianity, and finite nonlinearity

a simple discrete example

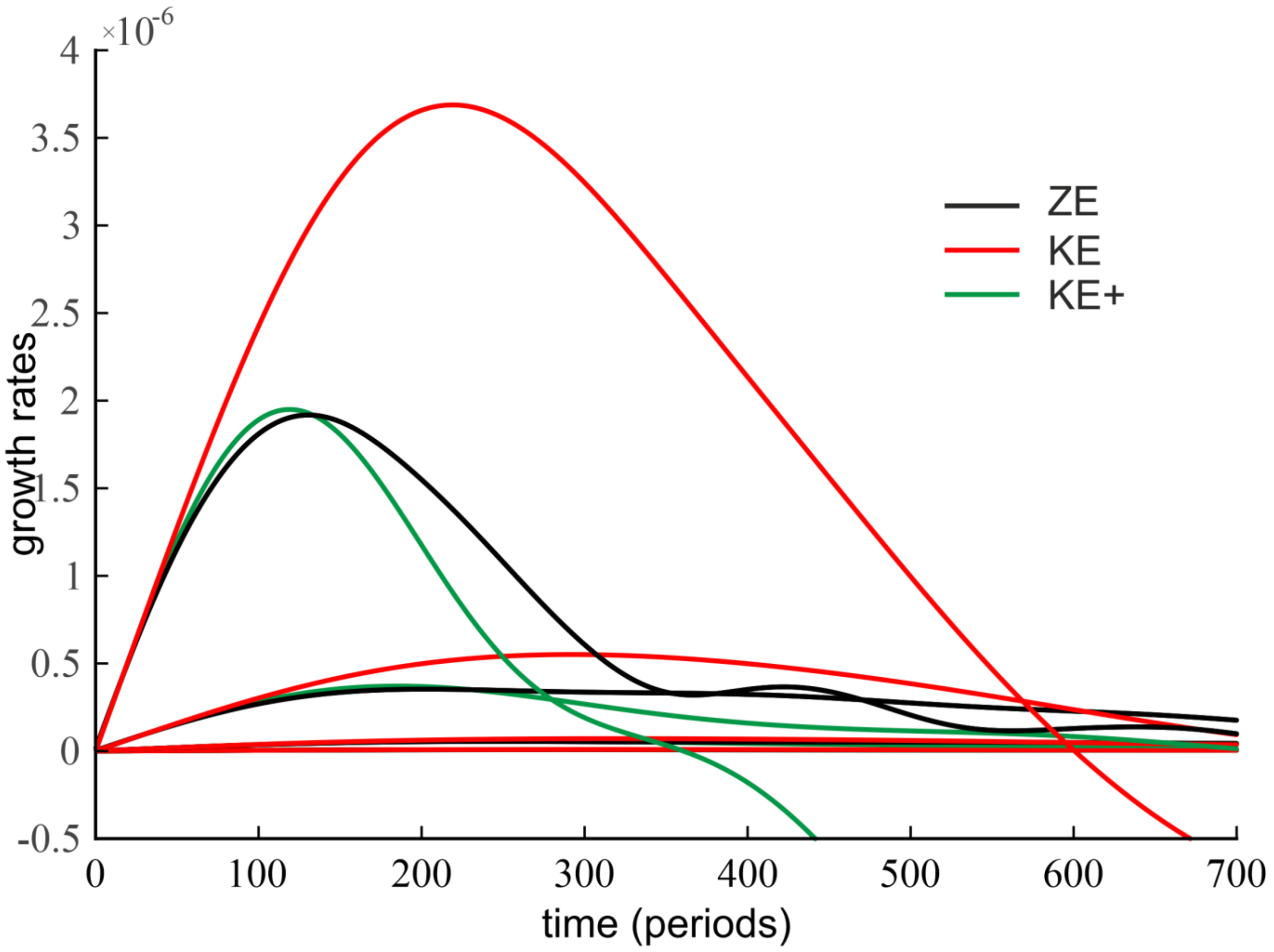




The longest harmonic of a discrete system is slowly growing due to nonlinear interactions. Total steepness is 0.05, non-Gaussianity is low, can we recreate the growth rate using the equation for correlators? (We solve the equation for correlators, using amplitudes obtained from the Zakharov equation with averaging over 10000 realisations, with and without the new terms, and then use correlators to calculate the growth rate)

scaling of growth rates

- Evolution of amplitudes is obtained with the Zakharov equation (10000 realisations)
- Growth rates are calculated using the equation for correlators, with and without $\hat{L}J$ term
- Growth rates obtained with the classic kinetics (red curves) tend to stick to the kinetic ε^6 scaling
- Account for finite non-Gaussianity ($\hat{L}J$ term, green curves) does not affect growth rates for very small ε, but for larger (still small) ε the scaling tends to ε^4
- The Zakharov equation growth rates are shown in black curves



Steepness ε=0.035, 0.05, 0.071 and 0.1

algorithm for the finite non-Gaussianity kinetics?

We need to create an algorithm for the solution of two linked systems of equations

$$\frac{\partial n_0}{\partial t} = 2\mathrm{Im}\int T_{0123} J_{0123} \delta_{0+1-2-3}\, \mathrm{d}\mathbf{k}_{123}, \qquad (6a)$$

$$\left(\frac{\partial}{\partial t} - \mathrm{i}\Delta\omega\right) J_{0123} = 2\mathrm{i} T_{0123} f_{0123} + 2\mathrm{i}\left(\hat{L}J\right)_{0123}, \qquad (6b)$$

where $f = n_2 n_3(n_0 + n_1) - n_0 n_1(n_2 + n_2)$ and the cumbersome expression for $\hat{L}J$ was given above. The two systems have N and $O(N^2)$ equations respectively, initial conditions should be specified as $n_j(0)$ and $J_{ijkl}(0)$.

For simplicity, we separate the two systems (each one is solved with the other one frozen, on one timestep). Effectively, this is Janssen (2003) approach.

The numerical solution is then obtained with two linked RKF timestepping algorithms, with tolerance 10^{-7} for amplitudes and 10^{-16} for correlators, and additional (small) limit on the maximal step size (at least 0.1 of the usual gKE step size).

further approximations

- Unfortunately, unlike in the standard gKE, such an algorithm requires all parallel processes to keep the current values of all correlators, broadcasting them multiple times per timestep. This overloads the interconnect and effectively ruins the computation
- To avoid this, an approximation in the spirit of Janssen (2003) approach is used again, this time for correlators: the value of $\hat{L}J$ is frozen at each timestep, and then the system for correlators can be integrated analytically (for one small timestep from 0 to t) as

$$J_{0123}(t) = 2T_{0123}\big(f_{0123}(0) + \hat{L}J(0)\big)G(\Delta\omega, t) + J_{0123}(0)e^{\mathrm{i}\Delta\omega t},$$

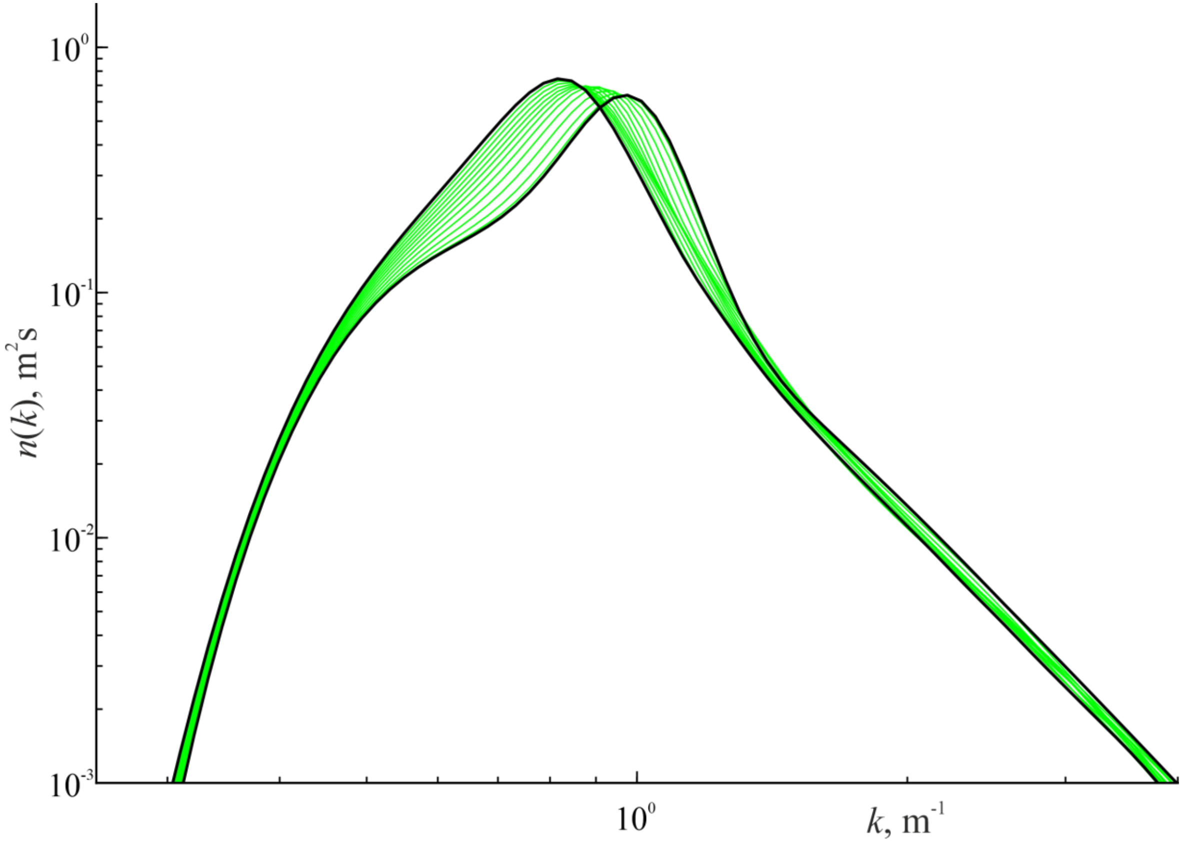
where $G(\Delta\omega, t) = \mathrm{i}\int_0^t e^{\mathrm{i}\Delta\omega(\tau - t)}\mathrm{d}\tau = \frac{1-\cos(\Delta\omega t)}{\Delta\omega} + \mathrm{i}\,\frac{\sin(\Delta\omega t)}{\Delta\omega}$ are Janssen's functions.

- The approach is justified if we can assume that correlators mostly depend on those correlators that are closer to resonance, that is larger and slowly changing ones
- Numerical experiments support this idea, provided that timestep is sufficiently small
- Parallel processes broadcast correlators to each other only once per timestep

Note: in the large time limit this approximation leads to the modified Hasselmann equation with two generalised functions, as $\lim_{t\to\infty} G(\Delta\omega, t) = \frac{P}{\Delta\omega} + \mathrm{i}\pi\delta(\Delta\omega)$

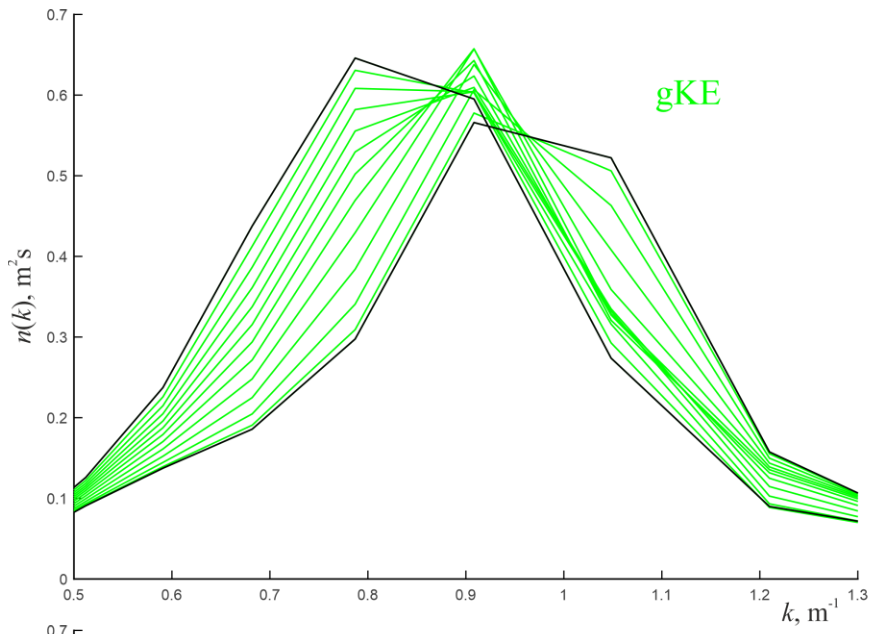
attempt at a numerical example

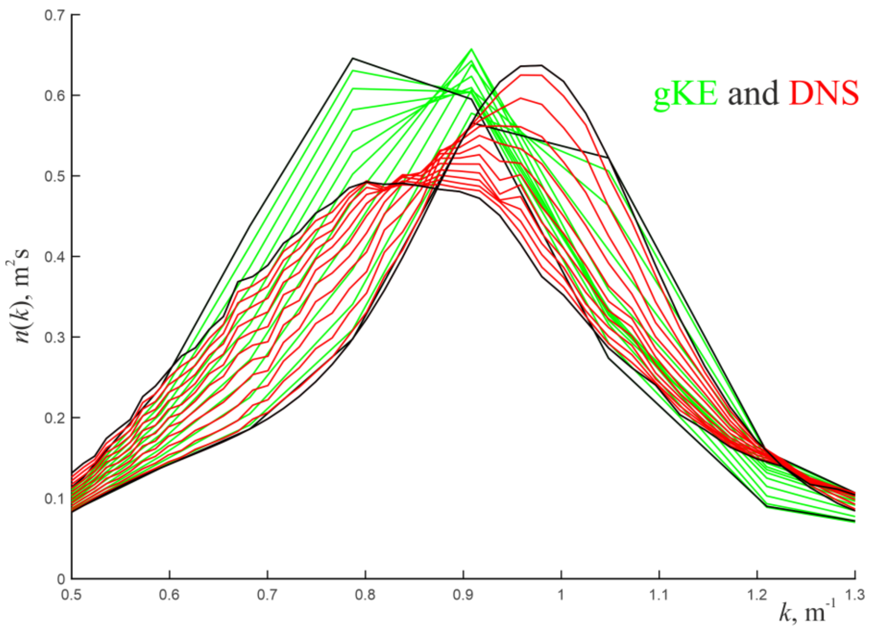
- As an example, consider the initial part of the evolution of Donelan's spectral shape corresponding to U/c=5.
- Figure shows first 1000 periods of evolution with weak forcing U/c=1.3 and dissipation in high frequencies.
- Grid has 101x31 harmonics in 160 degrees sector, with 190 million approximately resonant interactions ($\Delta\omega$ up to 0.01)
- This gKE simulation takes about 20 minutes on 128 processors
- No, for this example I can't simulate gKE+ (yet), but we can try a reduced resolution

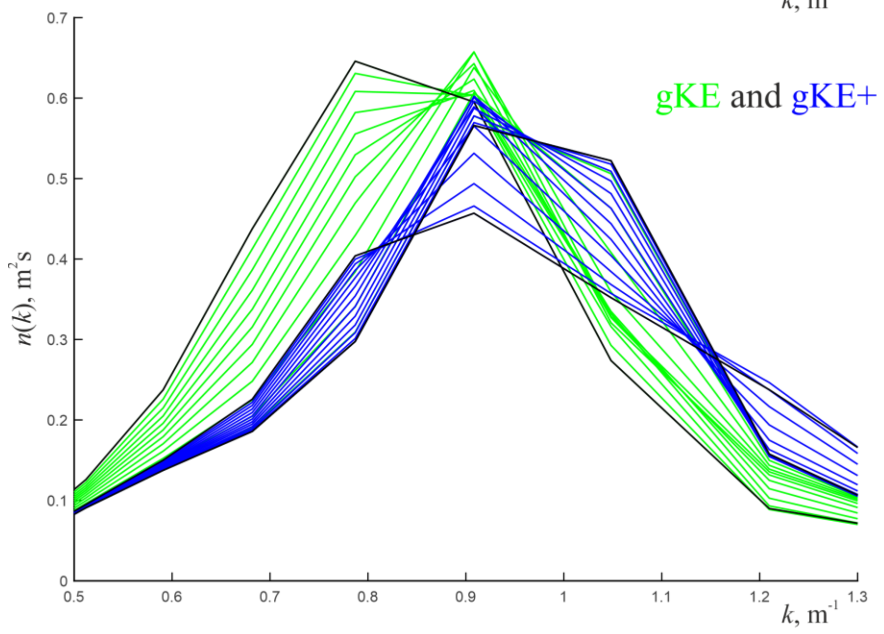


attempt at a numerical example

- We can take a low-resolution example, with 26x13 grid and 185000 interactions
- Evolution is for 200 periods only
- Solution for correlators is numerical, as analytical one not implemented yet
- Comparison with high-resolution DNS is also shown







conclusions

- This is work in progress. We are studying a generalisation of the generalised kinetic equation (gKE+) with the account for finite non-Gaussianity (and finite nonlinearity)
- Using a discrete wave system as a toy problem, we have found that the $\hat{L}J$ term is significant, as it changes the scaling of growth rates from ε^6 to ε^4 (above a certain, rather low, level of nonlinearity)
- In the general context, this means that all large growth rates tend to be reduced by finite non-Gaussianity effects
- All kinetic equations (both Hasselmann and gKE) completely neglect finite non-Gaussianity
- Although non-Gaussianity is weak, neglecting it violates the cornerstone principle of wave turbulence: equal play of weak nonlinearity and weak non-Gaussianity. Ignoring finite non-Gaussianity, we lose finite nonlinearity either
- We have shown that it is possible to simulate gKE+ numerically with a parallel algorithm, although the role of various approximations needs to be studied further
- gKE+ shows that the evolution depends on non-resonant interactions even for large times
- Are integral parameters (such as significant wave height) affected? Short answer: probably not.
- But spectral shapes are affected. There are, and there will be more in future, many applications that depend on spectral shapes