On extreme events in random-phase NLS: onset of MI and the length of the domain

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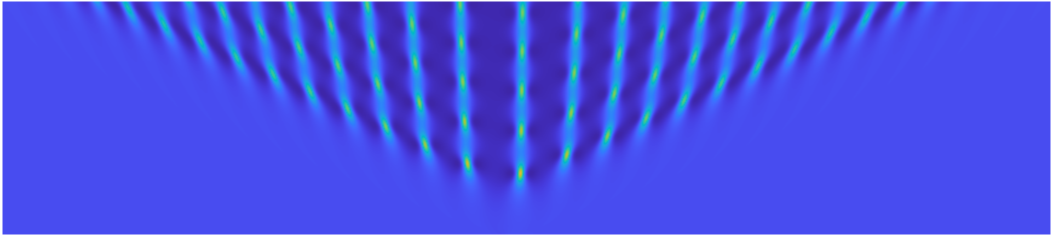
Classical Modulation Instability (MI)



$$i\partial_t U + p\Delta U + q|U|^2 U = 0, \quad U(t=0) = A \implies U(x,t) = Ae^{iqA^2 t}$$

$$i\partial_t u + p\Delta u + q|u|^2 u = 0, \quad u(t=0) = A + \delta_0(x) \implies u(t) = U(t) + \delta(x,t)$$

Inhomogeneity grows, then settles in a “stable” pattern. Observe the lengthscale L_c.

[Biondini and Mantzavinos, 2016]

Alber equation & generalised MI (gMI)

- A second moment theory, $R(x,x',t) = \mathrm{E}[u(x,t)\overline{u}(x',t)]$
- Assume initial quasi-homogeneity, $R_0 = \Gamma(x-x') + \varepsilon\rho_0(x,x')$
- Use gaussian moment closure, e.g.

$$E\Big[u(x,t)\bar{u}(x,t)u(x,t)\bar{u}(x',t)\Big] \approx 2E\Big[u(x,t)\bar{u}(x',t)\Big]E\Big[u(x,t)\bar{u}(x,t)\Big]$$

- Leads to closed equation for the inhomogeneity $\rho = \rho(x,x',t)$

$$i\partial_t\rho + p\left(\Delta_x - \Delta_{x'}\right)\rho + 2q\left[\Gamma(x-x') + \varepsilon\rho(x,x')\right]\left[\rho(x,x) - \rho(x',x')\right] = 0$$

- Inhomogeneity predicted to grow exponentially if

$$\exists k,\Omega \;:\; 1 + \omega_0 k_0^2 \int\limits_k \frac{P(k+\frac{X}{2}) - P(k-\frac{X}{2})}{\Omega + \frac{\omega_0}{4k_0^2}kX}dk = 0$$

where $P(k) = \mathcal{F}_{y\to k}[\Gamma(y)]$. [Alber, 1978, Ribal et al., 2013, Gramstad, 2017, Athanassoulis et al., 2020]

Monte Carlo on the onset of gMI

$$i\partial_t u + p\Delta u + q|u|^2 u = 0, \qquad u(x,0) = u_{hom}(x) + \delta_0(x)$$
$$u(-\tfrac{L}{2},t) = u(\tfrac{L}{2},t), \quad \partial_x u(-\tfrac{L}{2},t) = \partial_x u(\tfrac{L}{2},t),$$

$$u_{hom}(x) = C\sum_j A_j \sqrt{\tfrac{1}{2L} S(\tfrac{j}{L})}\, e^{2\pi i(\frac{jx}{L})}, \qquad A_j = \mathcal{N}(0,1) + i\mathcal{N}(0,1) \text{ iid complex normal RV.}$$

[Janssen, 2003, Athanassoulis and Gramstad, 2021]

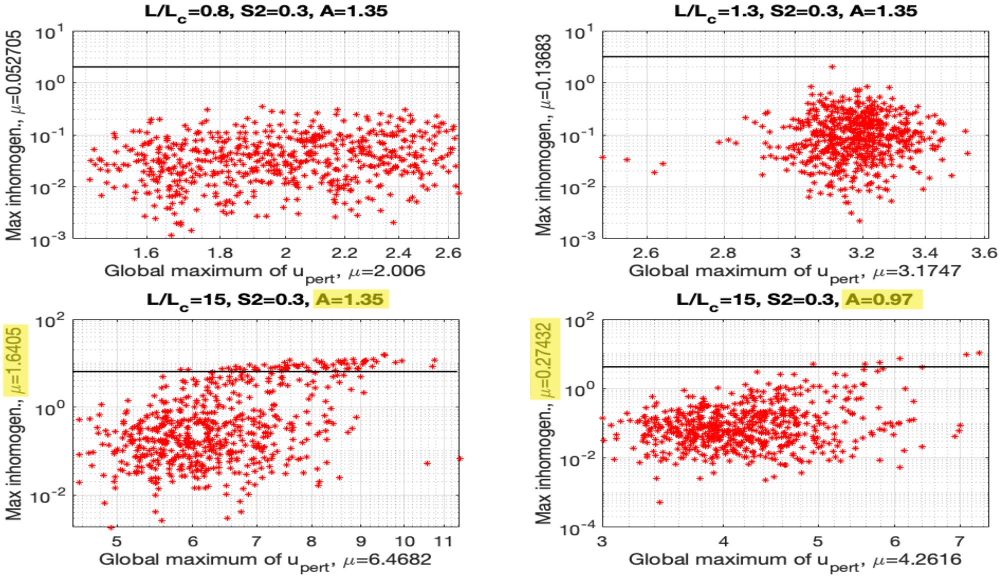
- $\delta_0(x)$ a small, localized perturbation
- Recall, for a plane wave of amplitude A on the torus of length L,

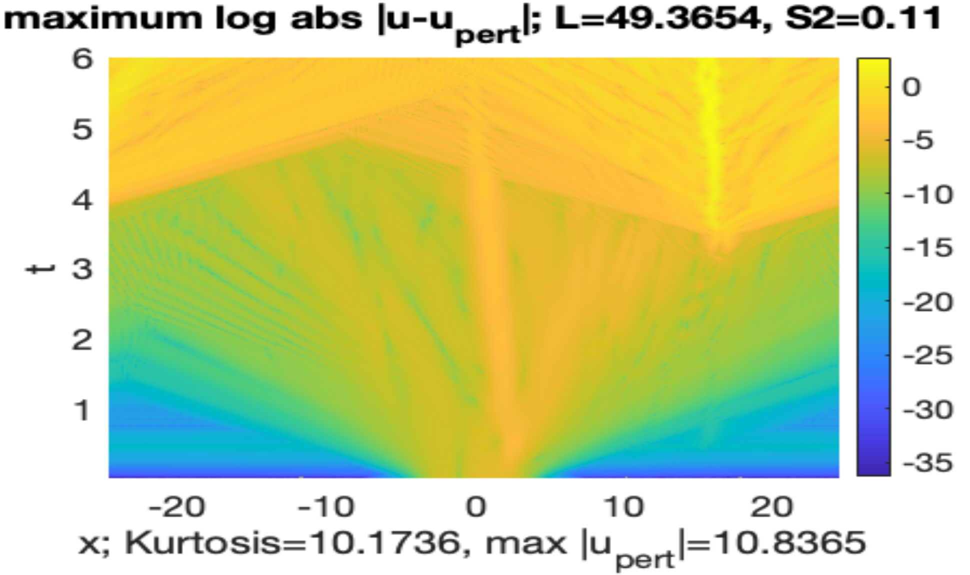
$$\omega_n^2 < 0 \iff L > \frac{2\pi|n|}{A}\sqrt{\frac{p}{2q}} = L_c = O(\lambda_0^2).$$

There is no MI for $L < L_c$. [Athanassoulis and Kyza, 2024]

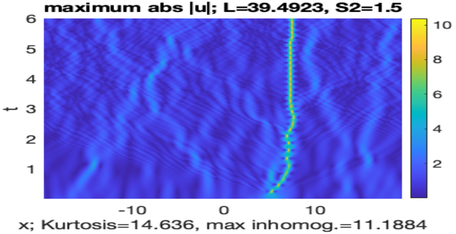
- $L = N \cdot L_c, \quad p = q = 1$
- "Amplitude": $rms|u_0| = A \iff \sqrt{\int\limits_{x=-L/2}^{L/2} |u_0(x)|^2 dx} = A \cdot L$
- NLS $+$ $u_{hom}(x,0) = u_{hom}(x)$ $+$ periodic BCs on $[-L/2, L/2]$
- NLS $+$ $u_{pert}(x,0) = u_{hom}(x) + \delta_0(x)$ $+$ periodic BCs on $[-L/2, L/2]$
- $\delta_0(x)$ is a localized bump with amplitude $\sim 5 \cdot 10^{-3} \cdot A$ $\rightsquigarrow$ highly correlated phases!
- Thus the inhomogeneity is $\delta(x,t) = u_{pert}(x,t) - u_{hom}(x,t)$
- Gaussian $P(k) = Cexp(-S2 \cdot k^2)$ & JONSWAP-like spectra
- $N = 300 - 900$; Solver: [Besse et al., 2021]

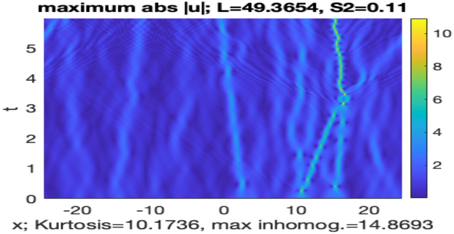
Some work in this direction: [Athanassoulis and Kyza, 2024, Athanassoulis, 2023]

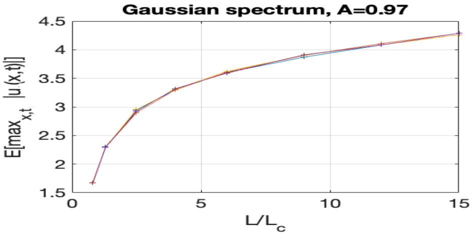

L/Lc=0.8, S2=0.3, A=1.35
Max inhomogen., μ=0.052705
Global maximum of upert, μ=2.006
L/Lc=1.3, S2=0.3, A=1.35
Max inhomogen., μ=0.13683
Global maximum of upert, μ=3.1747
L/Lc=15, S2=0.3, A=1.35
Max inhomogen., μ=1.6405
Global maximum of upert, μ=6.4682
L/Lc=15, S2=0.3, A=0.97
Max inhomogen., μ=0.27432
Global maximum of upert, μ=4.2616

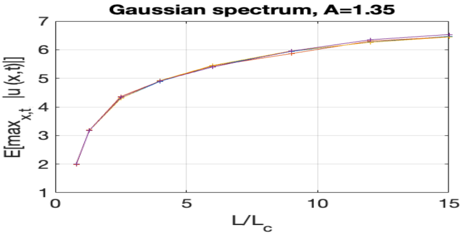

maximum log abs |u-u_pert|; L=49.3654, S2=0.11
t
x; Kurtosis=10.1736, max |u_pert|=10.8365
6
5
4
3
2
1
-20
-10
0
10
20
0
-5
-10
-15
-20
-25
-30
-35

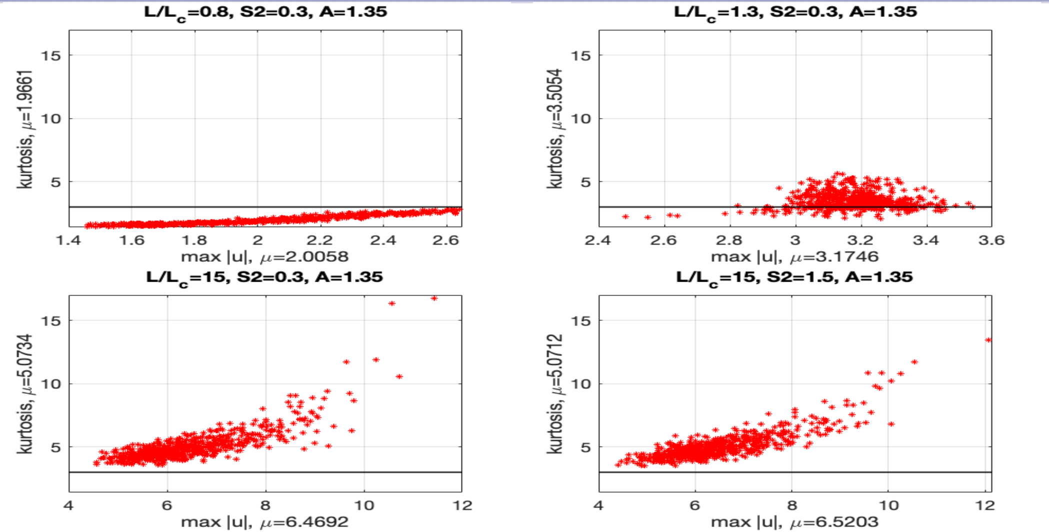
Nonlocal behaviour of maximum values in NLS











In fact, gaussian closures are rigorously justified only when $rms|u_0| = O(1/L)$ [Buckmaster et al., 2021]. This applies to all moment equations...

Conclusions

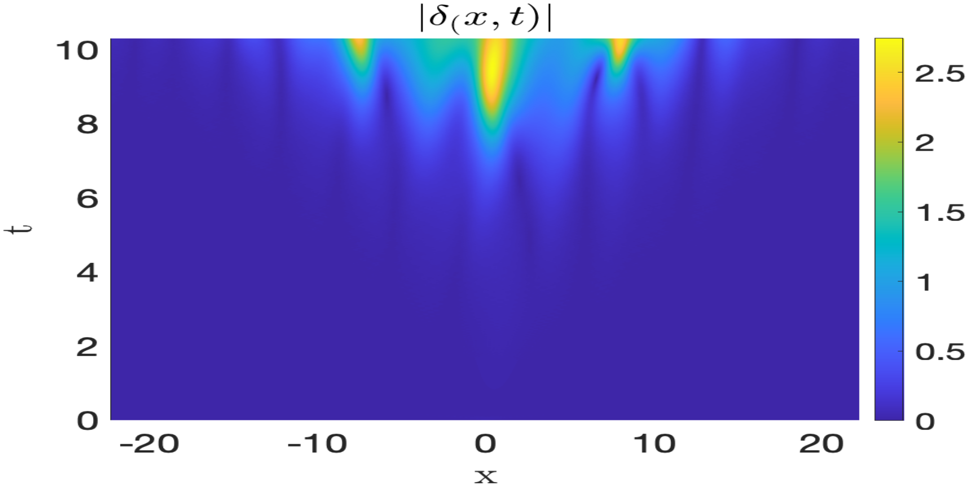
- Maxima in NLS wavefields are often due to breathers
  max values depend strongly on the length L

- **In all spectra tested, $L < L_c$ and $L > L_c$ are different problems**
  Recall that $L_c = O(\lambda_0^2)$ [Athanassoulis and Kyza, 2024]

- **gMI presents as inhomogeneities rapidly becoming $O(1)$.**
  The wavefield becomes much more chaotic [Athanassoulis, 2023]
  Qualitatively, it seems to require $7 - 20\ L_c$ to settle. [Ribal et al., 2013]
  No violent bifurcation in the statistics of extreme values *

- More broadly:
  Linear / gaussian statistics are problematic in gMI [Annenkov and Shrira, 2022]
  Sensitivity analysis in all parameters required.

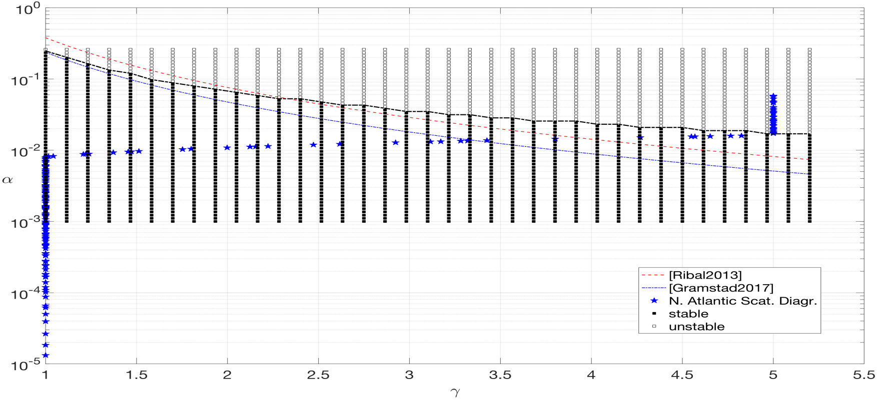
gMI → MI in the narrowband *limit*



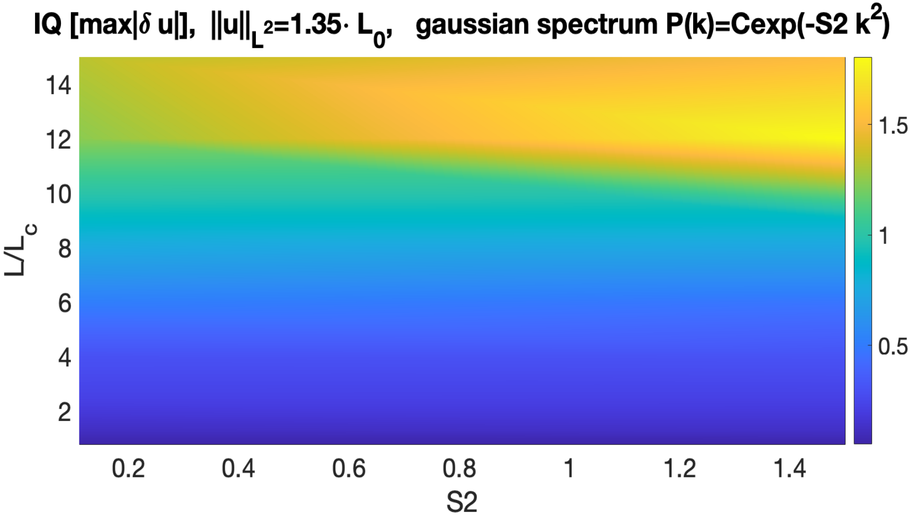
[Athanassoulis and Kyza, 2024]

Stability region for JONSWAP [Athanassoulis et al., 2020]

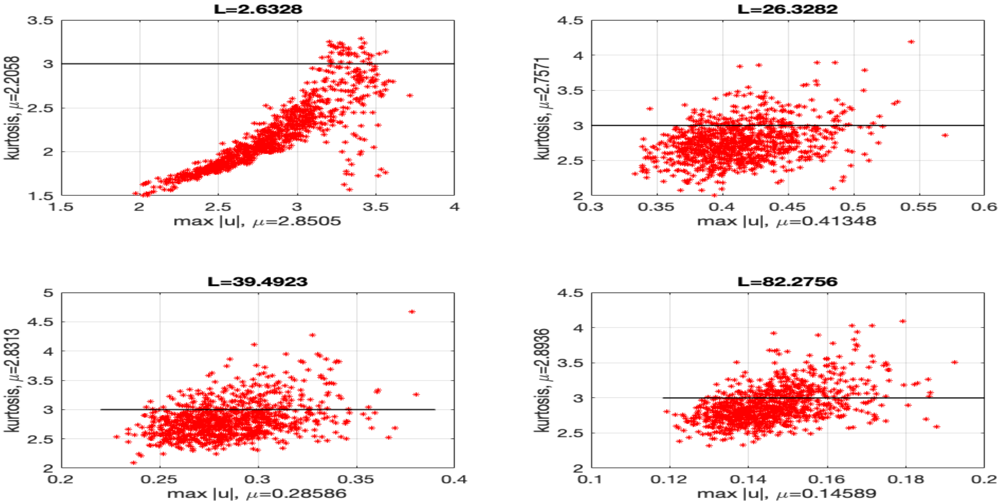
Unstable spectra: $\sim 0.2\%$ of the time.



North Atlantic Scatter Diagram data from [DNV-GL, 2017]


IQ [max|δ u|], ||u||L2=1.35· L0, gaussian spectrum P(k)=Cexp(-S2 k2)
L/Lc
S2
14
12
10
8
6
4
2
0.2
0.4
0.6
0.8
1
1.2
1.4
1.5
1
0.5

Convergent theory of gaussian closures: $|A| \sim 1/L$ (not water waves!!) [Buckmaster et al., 2021]



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